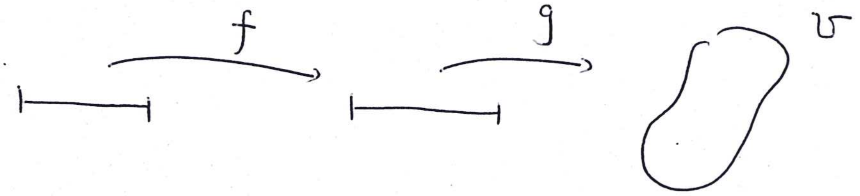
Appendix to lecture notes 9 and 10.

I. Suppose f and g are 2 functions. We have the following 3 possibilities.

(1)

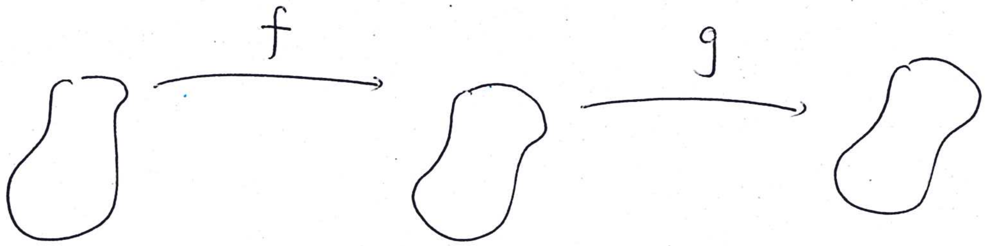


Chain rule in the case is

$$[g(f(t))]' = g'(f(t)) \cdot f'(t).$$

Here both "$'$" are derivatives with respect to single real variables.

(2)

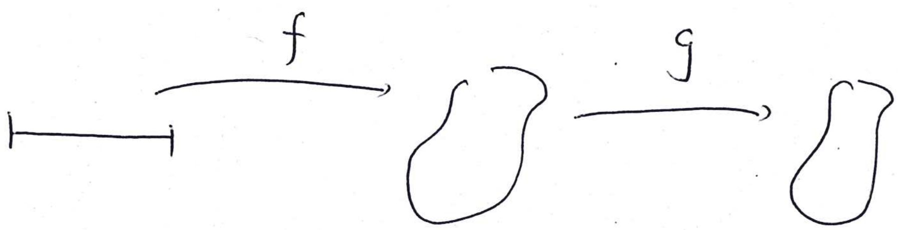


Chain rule is

$$(g(f(z)))' = g'(f(z))\, f'(z).$$

both "$'$" are derivatives with respect to single complex variables.

(3).



Chain rule.

$$\big(g(f(t))\big)' = g'(f(t)) \cdot f'(t)$$

Here "$'$" on g is complex derivative. "$'$" on f is derivative of single real variable.

In fact, $g(z) = u(x,y) + i v(x,y)$

$$f(t) = f_1(t) + i f_2(t)$$

$\therefore$ $g(f(t)) = u(f_1(t), f_2(t)) + i\, v(f_1(t), f_2(t))$

$\therefore$
$$\frac{d}{dt} g(f(t)) = \partial_x u\big|_{f_1,f_2} \cdot f_1' + \partial_y u\big|_{f_1,f_2} f_2' + i\, \partial_x v\big|_{f_1,f_2} f_1' + i\, \partial_y v\big|_{f_1,f_2} f_2'(t)$$

$$\overset{C\text{-}R}{=} \partial_x u\big|_{f_1,f_2} f_1' - \partial_x v\big|_{f_1,f_2} f_2' + i\, \partial_x v\big|_{f_1,f_2} f_1' + i\, \partial_x u\big|_{f_1,f_2} \cdot f_2'$$

$$= \partial_x u\big|_{f_1,f_2} \cdot (f_1' + i f_2') + i\, \partial_x v\big|_{f_1,f_2} \cdot (f_1' + i f_2')$$

$$= \left(\partial_x u\big|_{f_1,f_2} + i\, \partial_x v\big|_{f_1,f_2}\right) f'$$

$$= g'(f(t)) \cdot f'.$$

II. In lecture, we have shown if C has 2 one-one correspondences between intervals and C. and if 2 correspondences give same initial and ending points, then we have

$$\int_a^b f(\gamma_1(t))\,\gamma_1'(t)\,dt = \int_c^d f(\gamma_2(s))\,\gamma_2'(s)\,ds$$

Here $\gamma_1 : [a,b] \to C$

$\gamma_2 : [c,d] \to C$

are 2 one-one correspondences with

$\gamma_1(a) = \gamma_2(c)$, $\gamma_1(b) = \gamma_2(d)$.

In fact if C admits a one-one correspondence

$\gamma : [a,b] \to C$.

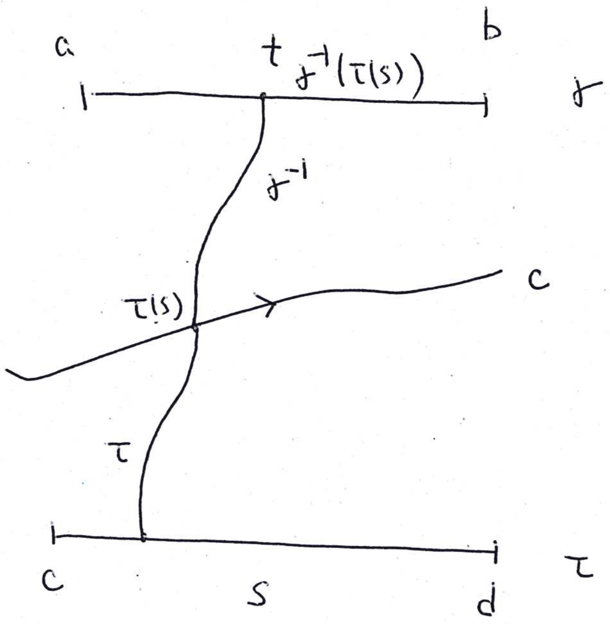
then for any parametrization of C, denoted by

$\tau : [c,d] \to C$.

we always have

$$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_c^d f(\tau(s))\,\tau'(s)\,ds,$$ provided

that $\gamma(a) = \tau(c)$, $\gamma(b) = \tau(d)$. τ is not necessary to be one-one.



only if γ is 1-1 can allow us define the function

$$\gamma^{-1} \circ \tau : [c, d] \to [a, b]$$

$$\therefore \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_c^d f\left(\gamma\left(\gamma^{-1}(\tau(s))\right)\right) \gamma'\left(\gamma^{-1}(\tau(s))\right) \cdot \left(\gamma^{-1}(\tau(s))\right)'\,ds$$

$$= \int_c^d f(\tau(s))\,\tau'(s)\,ds.$$

$\therefore$ if C admits a 1-1 correspondence between some interval and its self C.

then the contour integral

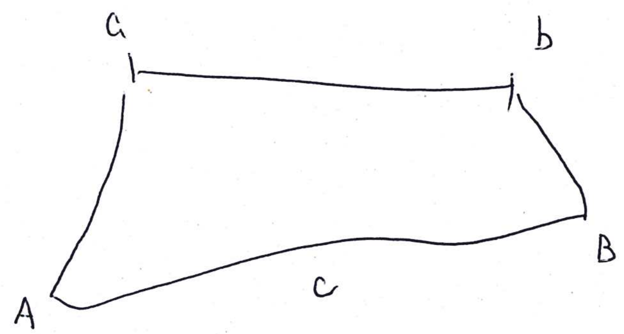
$\int_C f(z)\,dz$ is independent of parametrizations.

and only depend on direction of C.

In other words, directional curve C uniquely determines $\int_C f(z)\,dz$ if C has a one-one parametrization.

Q: what curves have 1-1 correspondence?

if $\gamma : [a,b] \to C$ sweeps out all points on C and γ is 1-1, see below



then by 1-1 assumption, for $t_1 \neq t_2$, it holds $\gamma(t_1) \neq \gamma(t_2)$. It implies that C has no self intersection. Therefore for C without self intersection, $\int_C f(z)\,dz$ is uniquely determined by its direction and independent of parametrizations. That is the 1-1 assumption for parametrization of C can be dropped in this case.